Indian Statistical Institute, Bangalore.
M. Math Back-paper Exam : Measure-theoretic Probability

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Max. points : 30. **Time Limit : 3 hours.**

Answer any three questions only but at least one question each from Part I and Part II needs to be answered.

Give complete proofs. Please cite/quote appropriate results from class or assignments properly. You are also allowed to use results from other problems in the question paper.

1 PART I

1. (a) State and prove the extended dominated convergence theorem by replacing convergence a.e. with convergence in measure. (**6**)

   (b) Let $(\Omega, \mathcal{F})$ be a measurable space and $f : \Omega \to \mathbb{R}$ be a measurable function such that $f(\Omega)$ is finite. Show that f is measurable iff $f = \sum_{i=1}^{k} a_i \mathbf{1}_{E_i}$ for $a_i \in \mathbb{R}$, disjoint $E_i \in \mathcal{F}$ such that $\Omega = \cup_{i=1}^{k} E_i$. (**4**)

2. Let $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ be a measure space and set $a_n := \mu(\{n\})$. Let $f : \mathbb{N} \to \mathbb{R}_+$. Show that $\sum_n f(n)a_n = \int f \mathrm{d}\mu$. (**10**)

3. Let μ be the restriction of the Lebesgue measure m on $\mathbb{R}^2$ to the σ-field $\mathcal{F} = \{A \times \mathbb{R} : A \in \mathcal{B}(\mathbb{R})\}$ of vertical strips. Define $\nu(A \times \mathbb{R}) = m(A \times (0,1))$. Does ν have a density w.r.t μ ? Which of the assumptions of the Radon-Nikodym theorem hold and which of them do not hold ? (**10**)

2 PART II

1. Let $X_i, i \geq 1$ be i.i.d. Bernoulli$(1/2)$ random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that $Y = \sum_{i=1}^{\infty} X_i 2^{-i}$ is a uniform $([0,1])$ random variable. (**10**)

2. Let $X, X_n, n \geq 1$ be random variables. Show that $X_n \xrightarrow{d} X$ iff $\mathbb{E}(f(X_n)) \to \mathbb{E}(f(X))$ for all bounded continuous functions f. (**10**)